Common-Deadline Lazy Bureaucrat Scheduling Problems

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Introduction

- **Scheduling problems:** A number of jobs to be performed by some employees.

- **Common studies:** To be as efficient as possible. The *employer’s* point of view.

- **Lazy bureaucrat:** To be as inefficient as possible. The *lazy employee’s* point of view. These are called *Lazy Beauracrat Scheduling Problems* (LBSP).

- **Some restrictions are needed to get out of degenerate cases.**

- **Similar considerations:** Shortest vs longest path problems.
Problem Definition

- A set $J$ of jobs $j_1, \ldots, j_n$, processing times (job lengths) $t_1, \ldots, t_n$, arrival times $a_1, \ldots, a_n$, deadlines $d_1, \ldots, d_n$.

- Jobs should be processed completely, if ever. We consider the case with no preemption.

- Hard deadlines. Processed jobs should be done so not before their arrival neither after their deadline. We call the interval $[a_i, d_i]$ the *window* for job $i$.

- Offline scheduling. We now the arrival times and deadlines beforehand.

- All the numbers are non-negative integers. Assume WLOG that there is a job arriving at time 0. Let $D = \max(d_i)$. 
More Definitions

• **Executable Job:** Job $j_i$ is *executable* at time $t$, iff it is not processed yet, and $a_i \leq t \leq d_i - t_i$.

• **Greedy Requirement:** At any time, the bureaucrat should work on an executable job, if there is any.

• The goal is to be as inefficient as possible. This is captured by any of the following objective functions that is to minimize.
Objective Functions

1. [min-time-spent]: Minimize the total amount of time spent working.

2. [min-weighted-sum]: Minimize the weighted sum of completed jobs.

3. [min-makespan]: Minimize the makespan, the maximum completion time of the jobs.

4. [min-number-of-jobs]: Minimize the total number of completed jobs.

Objective functions 1 and 4 are special cases of 2. If all jobs have the same arrival time, the objective functions 1 and 3 are equivalent.
Some Previous Results

Non-preemptive and multi-employee cases have also been studied. They are not listed here.

- LBSP is strongly NP-hard under all objective functions and is not approximable to within any fixed factor.

- LBSP with the same arrival times for all jobs, is weakly NP-hard, and can be solved by a pseudo-polynomial dynamic programming algorithm.

- LBSP with all the jobs having unit lengths, can be solved in polynomial time by the Latest Deadline First (LDF) scheduling policy.
Some Previous Results (continued)

- Assuming for each job $i$, $d_i - a_i < 2t_i$, LBSP can be solved in $O(nD \max(n, D))$ time.

- Even with a bound on $\delta$ (the ratio of the longest job to the shortest job), LBSP is strongly NP-hard. It cannot be approximated to within a factor of $\delta - \epsilon$, for any $\epsilon > 0$, unless $P = NP$.

- Given bounds on $R$ (the maximum ratio of window length to job length) and $\delta$, LBSP can be solved in $O(Dn^4 R \log \delta)$.

- Assuming $d_i - a_i < 2t_i$ for each job $i$ ($R \leq 2$), LBSP can be solved in $O(nD)$ time.
Our Problem — Notation

- **Common-Deadline LBSP (CD-LBSP)**: When deadlines for all jobs are the same ($D$).

- **CD-LBSP[objective-function]**: When considered with one of defined objective functions.

- **CD-LBSP[*]**: All/any of the objective functions.
Results

- CD-LBSP[*] is still NP-hard.

- CD-LBSP[\text{min-number-of-jobs}] is not approximable to within any fixed factor.

- There is a tight 2-approximation algorithm for CD-LBSP[\text{min-makespan}].

- CD-LBSP[*] is \textit{weakly} NP-hard: There exists a pseudo-polynomial time dynamic programming algorithm.
NP-Hardness

**Theorem 1** CD-LBSP[*] is NP-hard.

**Proof**

- Reduce the Subset Sum problem to this problem,

- Given: $S = \{x_1, \ldots, x_n\}$ of $n$ positive integers, where $\sum_{i=1}^{n} x_i = s$, and integer $b$ ($0 < b < s$),

- Is there any $T \subset S$, satisfying $\sum_{x \in T} x = b$?

- WLOG, we assume that $b \leq \lfloor \frac{s}{2} \rfloor$ and $x_i < b$ for all $i$.

- Construct an instance of CD-LBSP containing $n + 1$ jobs.
NP-Hardness (continued)

- All deadlines $D = 2s$,

- $\forall x_i \in S$, define job $j_i$ with $a_i = 0$, and $t_i = 2x_i$.

- Define $j_{n+1}$ with $a_{n+1} = 2b$ and $t_{n+1} = 2s - 2b - 1$.

- The employee can finish his work by time $2s$ or $2s - 1$,

- He can finish by $2s - 1$ iff he finds a solution for the Subset Sum problem.
Approximability

**Theorem 2** CD-LBSP[\text{min-number-of-jobs}] is not approximable to within any fixed factor \( \Delta > 1 \), unless \( P = NP \).

**Proof**

- Reduce Subset Sum problem again! This time to reach contradiction.

- Assume that there is an approximation algorithm with a fixed factor \( \Delta \).

- Let \( m = \lfloor \Delta \rfloor \) and \( D = b + m(n + 2)s \) (a huge number).
Approximabiity (continued)

- Construct an instance of CD-LBSP[min-number-of-jobs] containing the following jobs, all with deadline $D$:
  - $\forall x_i \in S$, define an element job $j_i$ having $a_i = 0$, and $t_i = x_i$,
  - Define one long job, $j_{n+1}$ with $a_{n+1} = b$ and $t_{n+1} = D - b$.
  - And define $m(n + 2) - 1$ extra jobs (lots of them), all having arrival times $b$, and processing times $s$.

- The bureaucrat wants to do as few jobs as possible: He should process the long job, to avoid too many extra jobs,

- So he can process as few as $n + 1$ jobs iff he finds a solution for the Subset Sum problem.
Approximablity (continued)

- The hypothetical $\Delta$-approximation would produce at most $m(n + 1)$ jobs.

- There are $m(n + 2) - 1$ extra jobs, that all can be processed if the long job is not.

- So the approximation algorithm is forced to produce the optimal solution $\Rightarrow$ solution for the Subset Sum problem $\Rightarrow P = NP$.

\[\square\]

**Corollary 1** CD-LBSP[min-weighted-sum] is not approximable to within any fixed factor $\Delta > 1$, unless $P = NP$. 
Approximation

**Theorem 3** The Shortest Job First (SJF) scheduling policy is a 2-approximation algorithm for CD-LBSP[\text{min-makespan}] and this bound is tight.

**Proof**

- Let $\sigma_{OPT}$ be an optimal solution and $\sigma$ be the schedule which the SJF policy has generated, and $OPT$ and $SJF$ be their makespans respectively,

- We show that $SJF - OPT < OPT$,

- WLOG suppose that $j_1, \ldots, j_k$ are the jobs processed in $\sigma$ in that order,

- Let $j_q \in \sigma$ be the job satisfying $start_q(\sigma) < OPT \leq finish_q(\sigma)$,
Approximation (continued)

- We know that \( a_i < OPT \) for all jobs,

- SJF policy forces that \( t_{q+1} \leq t_{q+2} \leq \ldots \leq t_k \),

- Greedy requirement forces that \( j_i \in \sigma_{OPT} \) for all \( q + 1 \leq i \leq k \)

- If \( j_q \in \sigma_{OPT} \) then \( SJF - OPT < \sum_{i=q}^{k} t_i \leq OPT \), and we are done,

- Otherwise, there exists some job in \( \sigma_{OPT} \) that is arrived before \( j_q \), and not shorter than \( j_q \), which is not processed in \( \sigma \), call it \( j_p \),

- So we have

\[
SJF - OPT < t_q + \sum_{i=q+1}^{k} t_i \leq t_p + \sum_{i=q+1}^{k} t_i \leq OPT.
\]
Tightness

- Given $n$ and $0 < \epsilon < 1$, we construct an instance of CD-LBSP with $n$ jobs, which SJF does no better than $2 - \epsilon$.

- All jobs have $a_i = 0$ and $d_i = D$, where 
  $$D = n - 3 + 2L - 1 = n + 2L - 4$$
  with $L = 2n/\epsilon$:

- $t_1 = L - 1$, $t_2 = L$, $t_3 = L + 1$, $t_i = 1$ ($4 \leq i \leq n$),

- OPT would process all unit jobs and then $j_3$, having makespan
  $$OPT = n - 3 + L + 1 = n + L - 2,$$

- SJF would process $j_4, \ldots, j_n$, $j_1$, and $j_2$ having makespan
  $$SJF = n - 3 + L - 1 + L = n + 2L - 4.$$

- With some magical math, we would have $\frac{SJF}{OPT} > 2 - \epsilon$. □
Pseudo-Polynomial Time Algorithms

- Assume that the jobs are numbered in order of their arrival times,

- Let $T_i = \{j_i, \ldots, j_n\}$, $T_{i,k} = \{j_i, \ldots, j_k\}$,

- We can assume that in the optimal schedule, the consecutive jobs are performed in order of their arrival times,

Lemma 1 For a given $(T_i, \alpha, U)$, we can find an optimal schedule without any gaps from some jobs in $T_i$, and up to time $\alpha$, so that all jobs in $U$ appear in the schedule, if any such schedule exists, in time $O(n\alpha)$.

Proof It can be solved much like the binary knapsack problem. □
Pseudo-Polynomial Time Algorithms (continued)

Theorem 4 CD-LBSP[*] is weakly NP-hard.

Proof

- Let $P_i$ be the subproblem of scheduling the jobs in $T_i$. $P_1$ is the original problem,

- Let $\alpha$ be the first rest time in an optimal schedule $\sigma$ for $P_i$,

- So the schedule can be broken into two independent subschedules, one before $\alpha$, and one after $\alpha$,

- For $\alpha$ to be a rest time, there are some jobs forced to be in the first schedule,
Pseudo-Polynomial Time Algorithms (continued)

- The first subschedule can be found by applying the lemma from previous page,

- The second one is the optimal solution to subproblem $P_k$, where $j_k$ is the first job arriving after $\alpha$.

- All we need to do is to search for $\alpha$.

- This all takes time $O(n^2 D^2)$ to solve $P_1$.

- So CD-LBSP[*] is weakly NP-hard.
Conclusion

We studied a new class of the Lazy Bureaucrat Scheduling Problems (LBSP), called common-deadline LBSP, where the deadlines of all jobs are the same. We proved that this problem is still NP-hard under all four pre-defined objective functions. We also showed that this problem is not approximable to within any fixed factor in cases of [min-weighted-sum] and [min-number-of-jobs] objective functions. The problem is shown to have a tight 2-approximation algorithm under [min-makespan]. But, it is still open whether it is approximable under [min-time-spent]. In the rest of the paper, we presented pseudo-polynomial time dynamic programming algorithms for this problem under all objective functions. Further work on this problem is underway.

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References


